Week 11:

Faraday's law of induction
and
Inductance

Transmission of Power

Because $P = IV$, the same amount of energy can be transported either at high currents and low potential differences or at low currents and high potential differences. It is economical to use a high voltage and a low current to minimize the $\boldsymbol{P = I^2 r}$ loss in transmission lines when electric power is transmitted over great distances.

Transmitted power: $P_{tr} = V \cdot I$, $(I = V/R)$

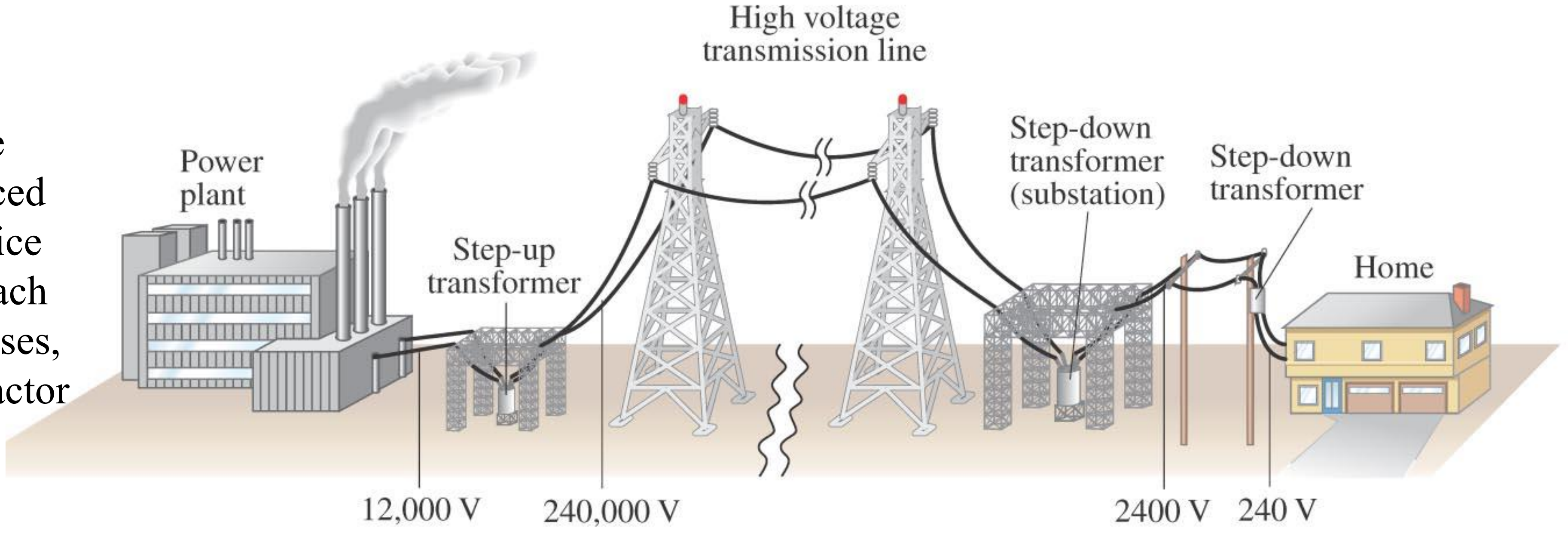
Losses: $p_{loss} = I^2 \cdot r = P_{tr}^2 \cdot r / V^2$; $r = \rho \frac{l}{A}$

- High V
- shorter $\boldsymbol{l}$
- good conductor (copper, alu)

Minimize power loss
$\boldsymbol{P = I^2 r}$

Copper wire is very expensive, so it is cheaper to use high-resistance r wire (having a small cross-sectional area A). Therefore, in $P = I^2 r$, **the resistance of the wire $\boldsymbol{r}$** is fixed at a relatively high value for economic considerations.
The $P = I^2 r$ loss can be reduced by keeping the current I as low as possible, which means transferring the energy at a high voltage.

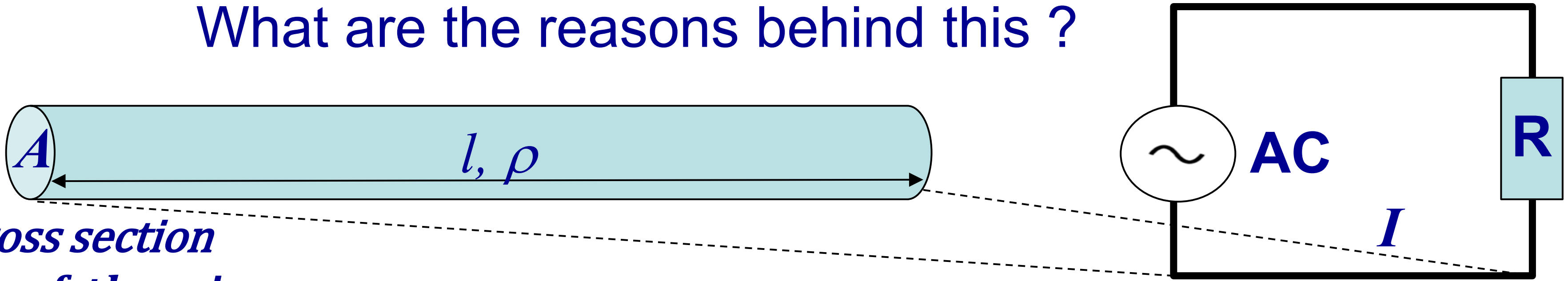
At the destination of the energy, the potential difference is usually reduced to 4 kV and then to 240 V by a device called a **transformer**. Of course, each time the potential difference decreases, the current increases by the same factor and the power remains the same.



AC at home

In **CH**: V_0=**240** V, f= 50 Hz; in **USA**: V_0=**115** V, f=60 Hz

What are the reasons behind this ?



A^{wire} = *wire cross section*
M_{wire} = *mass of the wire*
ρ = *density of the wire*

$$P_0 = VI \Rightarrow \frac{I_{CH}}{I_{US}} \sim 1/2 \Rightarrow \left(I_{max} \propto A^{wire}\right) \Rightarrow \frac{A_{CH}}{A_{US}} = 1/2$$

$$M_{wire} = \rho A l \Rightarrow \frac{M_{US}}{M_{CH}} = 2$$

CH is twice more economic !

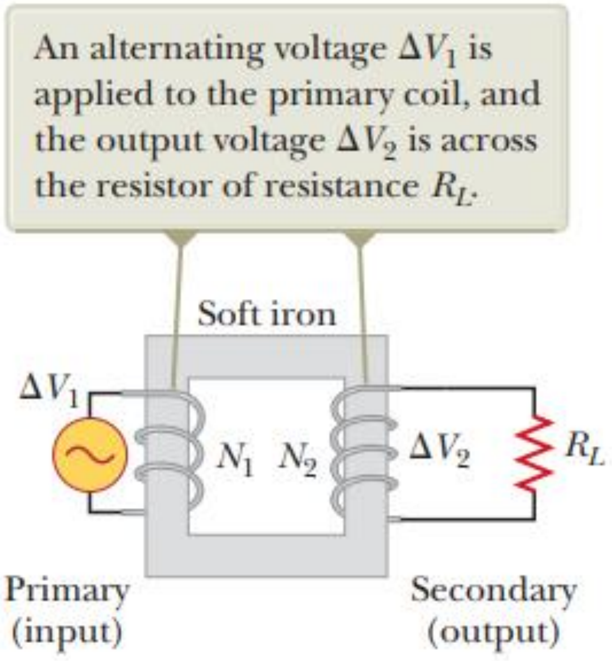
How to change V_0 along the power lines distribution?

Transmission and use of electric power

The **AC transformer** consists of two coils of wire wound around a core of iron. The coil on the left, which is connected to the input alternating-voltage source and has N_1 **turns**, is called **the *primary winding*** (or the primary). The coil on the right, consisting of N_2 **turns** and connected to a load resistor R_L , is called the ***secondary winding*** (or the secondary). The purposes of the iron core are to increase the magnetic flux through the coil and to provide a medium in which nearly all the magnetic field lines through one coil pass through the other coil.

Transformers work only if the current is changing;

this is one reason why electricity is transmitted as *AC*.



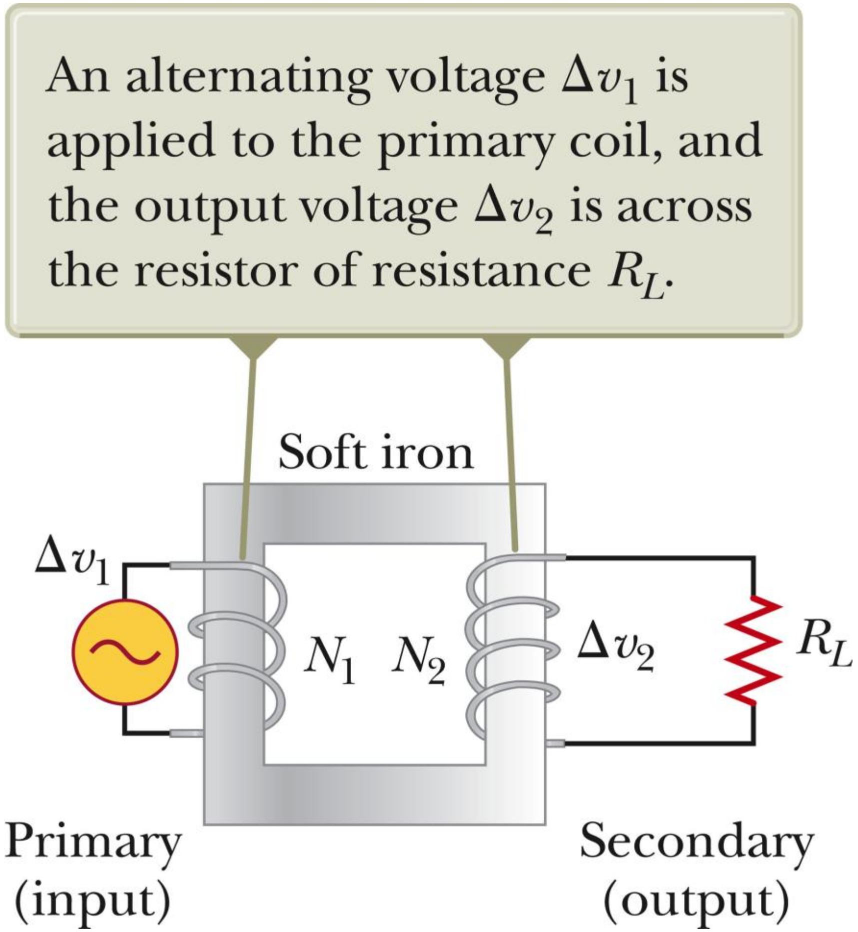
Transformers

An **AC transformer** consists of two coils of wire wound around a core of iron.

The side connected to the input AC voltage source is called the *primary* and has N_1 turns.

The other side, called the secondary, is connected to a resistor and has N_2 turns.

The core is used to increase the magnetic flux and to provide a medium for the flux to pass from one coil to the other.



Transformers – Step-up and Step-down

Eddy-current losses are minimized by using a laminated core.

Assume an ideal transformer

- One in which the energy losses in the windings and the core are zero.
  - Typical transformers have power efficiencies of 90% to 99%.

The voltages are related by

$$\Delta v_2 = \frac{N_2}{N_1}\Delta v_1 \Rightarrow \boxed{\frac{\Delta v_1}{\Delta v_2} = \frac{N_1}{N_2}}$$

The ratio of the *emfs* is equal to the ratio of the number of turns in each coil.

When $N_2 > N_1$, the transformer is referred to as a step-up transformer.

When $N_2 < N_1$, the transformer is referred to as a step-down transformer.

The power input into the primary equals the power output at the secondary.

- $I_1\Delta V_1 = I_2\Delta V_2$

  $P = IV = const;$

This is true for an ideal transformer for which we can neglect the internal resistance

The equivalent resistance of the load resistance when viewed from the primary is

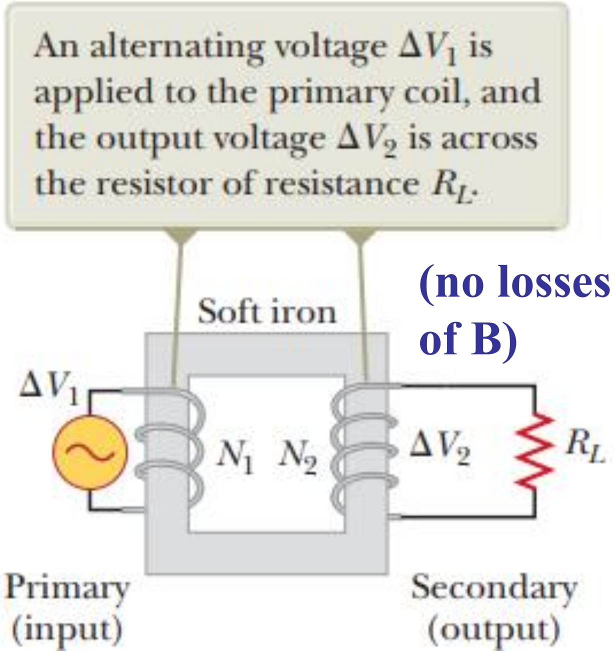
$$R_{eq} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

In the primary,

$$\Delta v_1 = -N_1\frac{d\Phi_B}{dt}$$

The rate of change of the flux is the same for both coils (and in each loop). The voltage across the secondary is

$$\Delta v_2 = -N_2\frac{d\Phi_B}{dt}$$



(no losses of B)

Transformers Summary

$$\frac{V_p}{V_s} = \frac{N_p}{N_s}; \quad \frac{I_s}{I_p} = \frac{N_p}{N_s}.$$

$N_s > N_p$ – **step-up** transformer:

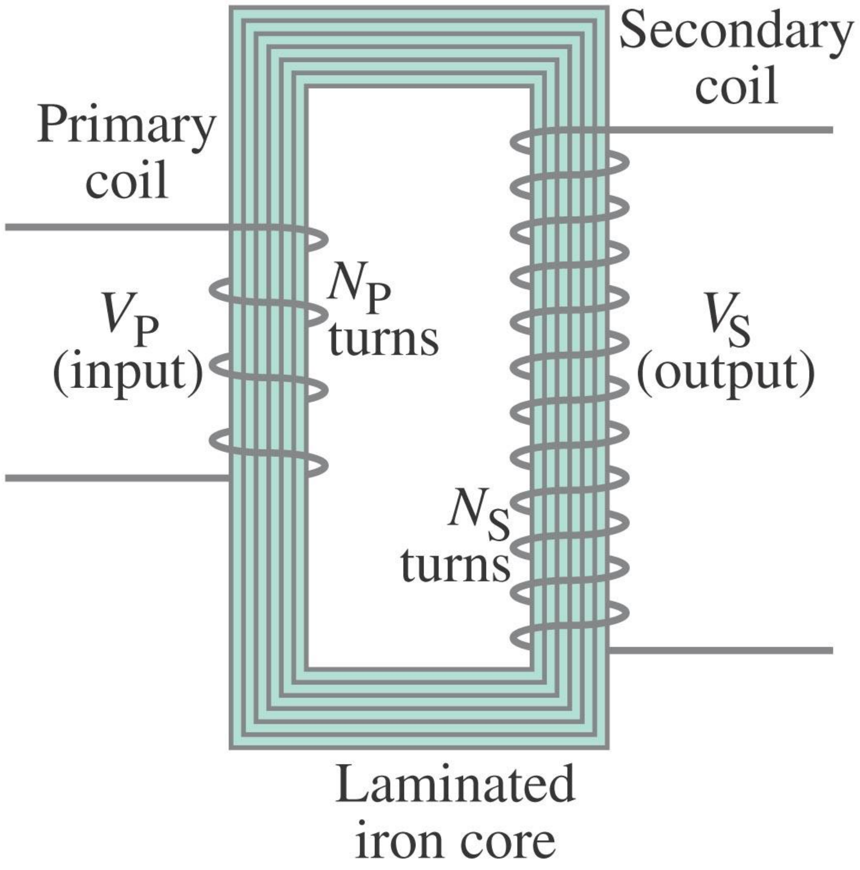
$$V_s > V_p$$

$$I_s < I_p$$

$N_s < N_p$ – **step-down** transformer:

$$V_s < V_p$$

$$I_s > I_p$$



Inductance

Self-inductance

- A time-varying current in a circuit produces an induced emf opposing the emf that initially set up the time-varying current.
  - Basis of the electrical circuit element called an *inductor*
- Energy is stored in the magnetic field of an inductor.
- There is an energy density associated with the magnetic field.

Mutual induction

- An emf is induced in a coil as a result of a changing magnetic flux produced by a second coil.

Circuits may contain inductors as well as resistors and capacitors.

Joseph Henry

Some Terminology

Use *emf* and *current* when they are caused by batteries or other sources.

Use *induced emf* and *induced current* when they are caused by changing magnetic fields.

When dealing with problems in electromagnetism, it is important to distinguish between the two situations.

Self-Inductance

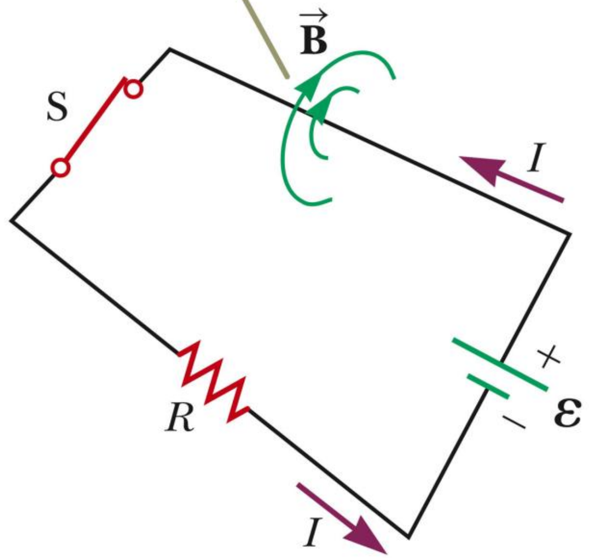
When the switch is closed, the current does not immediately reach its maximum value.

Faraday's law of electromagnetic induction can be used to describe the effect.

As the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time.

This increasing flux creates an **induced *emf*** in the circuit.

After the switch is closed, the current produces a magnetic flux through the area enclosed by the loop. As the current increases toward its equilibrium value, this magnetic flux changes in time and induces an emf in the loop.



The direction of the induced *emf* is such that it would cause an induced current in the loop which would establish a magnetic field opposing the change in the original magnetic field.

The direction of the induced *emf* is opposite the direction of the *emf* of the battery.

This results in a gradual increase in the current to its final equilibrium value.

This effect is called **self-inductance.**

- Because the changing flux through the circuit and the resultant induced *emf* arise from the circuit itself.

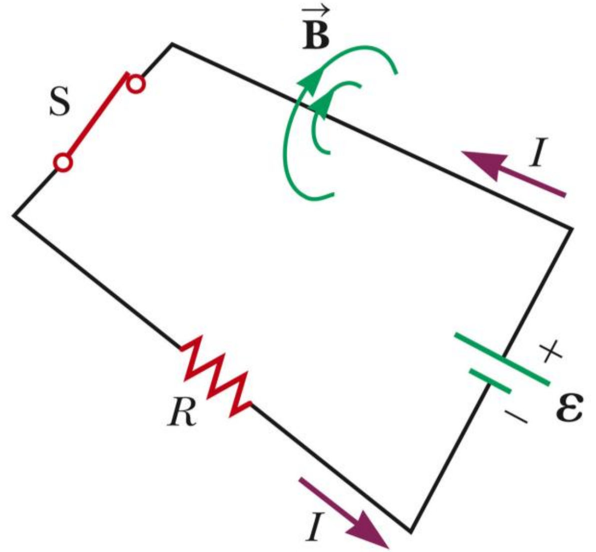
The emf ε_L is called a **self-induced *emf.***

Self-Inductance, Equations

An induced *emf* is always proportional to the time rate of change of the current.

- The *emf* is proportional to the flux, which is proportional to the field and the field is proportional to the current.

$$\varepsilon_L = -L\frac{d\mathrm{I}}{dt} = -Nd\Phi_B/dt$$



L is a constant of proportionality called the **inductance** of the coil.

- It depends on the geometry of the coil and other physical characteristics.

Combining this expression with Faraday's law $\varepsilon_L = -Nd\Phi_B/dt$

Inductance of a Coil

A closely spaced coil of N turns carrying current *I* has an inductance of

$$L = \frac{N\Phi_B}{I} = -\frac{\varepsilon_L}{dI/dt}$$

The inductance is a measure of the opposition to a change in current.

Inductance Units

The SI unit of inductance is the **henry** (H)

$$1\mathrm{H} = 1\frac{\mathrm{V}\cdot\mathrm{s}}{\mathrm{A}}$$

Named for Joseph Henry

Self-Inductance of a Solenoid

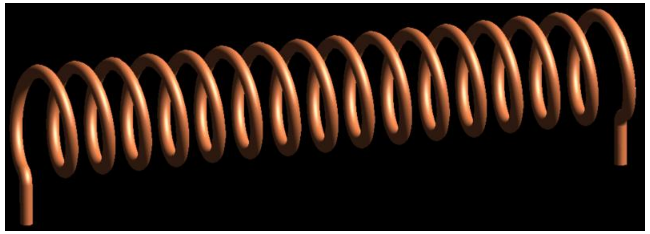
- **What *emf* $\mathcal{E}$ is generated in a solenoid with N turns, area A, for a certain rate of changing of current dI/dt througth the solenoid itself?**

Assume a uniformly wound solenoid having *N* turns and length ℓ.

- Assume ℓ is much greater than the radius of the solenoid.

The flux through each turn of area *A* is

$$\Phi_B = BA = \mu_o n \, \mathrm{I} \, A = \mu_o \frac{N}{\ell} \mathrm{I} \, A$$



N turns total in length ℓ: $N/\ell = n$ turns per meter. Cross-section A

The total flux is $N\Phi_\mathrm{B}$ => $$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -\frac{\mu_0 N^2 A}{\ell}\frac{dI}{dt}$$

The inductance is

$$L = \frac{N\Phi_B}{\mathrm{I}} = \frac{\mu_o N^2 A}{\ell} = \mu_o n^2 V$$

*where **$V = \ell A$ is the volume of the solenoid***

This shows that *L* depends on the geometry of the object.

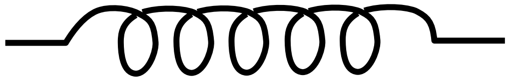
Definition of Self Inductance (summary)

For any shape conductor with changing current in a magnetic field there is an induced *emf* $\mathcal{E}$ proportional to the rate of change of current and (sign) opposing the change:

$$\mathcal{E} = -L\frac{dI}{dt}$$

$$L = \frac{N\Phi_B}{I}$$

***L*- self inductance**

***L* =** 

Unit: henrys: H= V·A^{-1}·s

For a solenoid:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -\frac{\mu_0 N^2 A}{\ell}\frac{dI}{dt}.$$

$$L = \frac{\mu_0 N^2 A}{\ell}$$

- **Self-inductance is a geometrical characteristic of a conductor**

Mutual Inductance

The magnetic flux through the area enclosed by a circuit often varies with time because of time-varying currents in nearby circuits.

This process is known as ***mutual induction*** because it depends on the interaction of two circuits.

The current in coil 1 sets up a magnetic field.

Some of the magnetic field lines pass through coil 2.

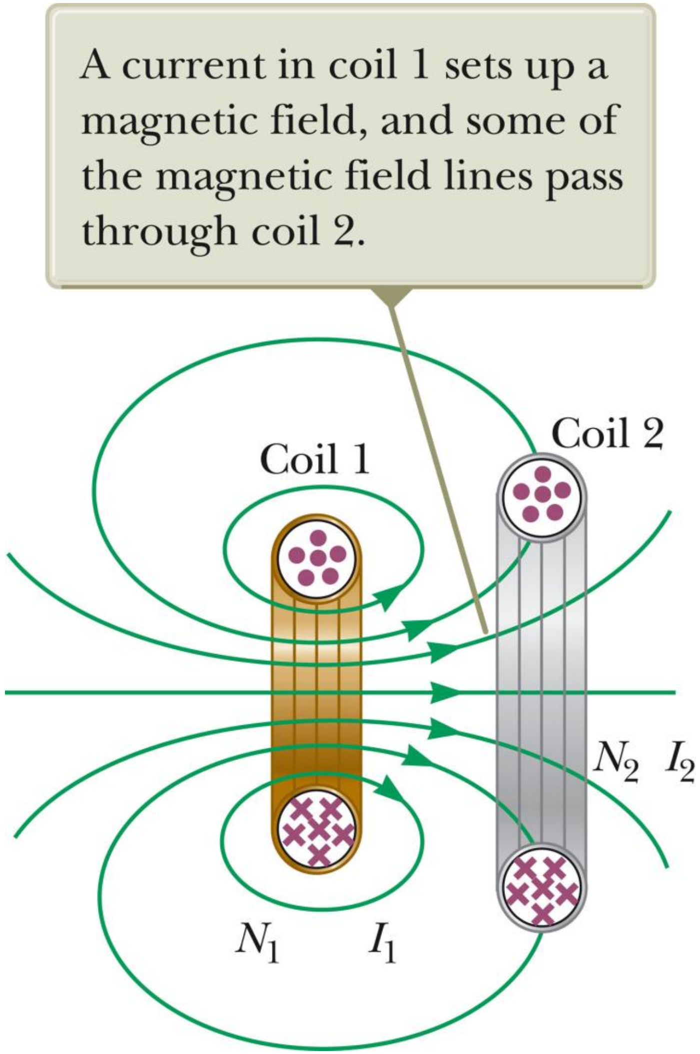
Coil 1 has a current I_1 and N_1 turns.

Coil 2 has N_2 turns.

The **mutual inductance** M_{12} of coil 2 with respect to coil 1 is

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1}$$

Mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other.



Induced *emf* in Mutual Inductance

If current I_1 varies with time, the emf induced by coil 1 in coil 2 is

$$\varepsilon_2 = -N_2 \frac{d\Phi_{12}}{dt} = -M_{12} \frac{d\mathrm{I}_1}{dt}$$

If the current is in coil 2, there is a mutual inductance M_{21}.

If current 2 varies with time, the emf induced by coil 2 in coil 1 is

$$\varepsilon_1 = -M_{21} \frac{d\mathrm{I}_2}{dt}$$

In mutual induction, the *emf* induced in one coil is always proportional to the rate at which the current in the other coil is changing.

The mutual inductance in one coil is equal to the mutual inductance in the other coil.

- $M_{12} = M_{21} = M$ (We will not demonstrate it for a general case; **BUT IT IS VALID IN GENERAL !**)

The induced emf's can be expressed as

$$\varepsilon_1 = -M \frac{d\mathrm{I}_2}{dt} \quad \text{and} \quad \varepsilon_2 = -M \frac{d\mathrm{I}_1}{dt}$$

Mutual Inductance for a transformer

A transformer: when the current I_1 in coil 1 changes, it gives rise to an emf $\mathcal{E}_2$ in coil 2.

The mutual inductance M_{21}

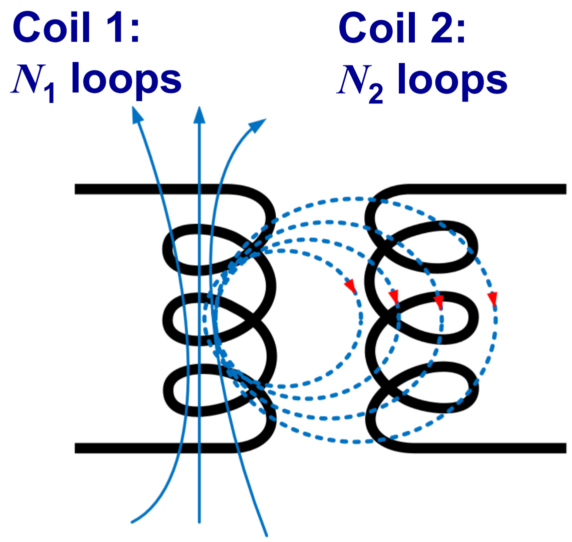
$$M_{21} = -\frac{\varepsilon_1}{d\mathrm{I}_2/dt} \quad \text{and} \quad M_{12} = -\frac{\varepsilon_2}{d\mathrm{I}_1/dt}$$

If no losses of magnetic field lines (transformer with iron core):

See slide 10.9 for a transformer

$$\varepsilon_1 = \varepsilon_2 \frac{N_1}{N_2}; \text{ and } I_2 = I_1 \frac{N_1}{N_2}; \Rightarrow$$

$$\Rightarrow \quad M_{21} = -\frac{\varepsilon_1}{d\mathrm{I}_2/dt} = -\frac{\varepsilon_2 (N_1/N_2)}{d\mathrm{I}_1/dt\,(N_1/N_2)} = M_{12}$$



$$M = M_{21} = M_{12}$$

- **Determined by the geometries and mutual position of the coils.**

Mutual Inductance (transformer with two solenoids and a magnetic core)

Ideal transformer with core $\mu \gg \mu_0$: (no losses) =>
the same flux *per a turn* in both coils.

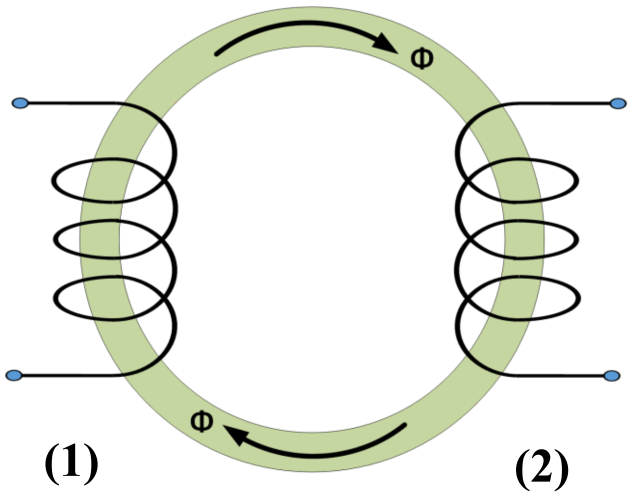
Φ_{21} Magnetic flux caused by the current in coil 2 and passing through coil 1

B_2

$$M_{21} = N_1 \frac{d\Phi_{21}}{dI_2} = N_1 \frac{d}{dI_2}\left(A \frac{N_2}{l} \mu I_2\right)$$

$$M_{21} = N_2 N_1 \mu \frac{A}{l} = M_{12} \qquad L = \frac{\mu N^2 A}{l}$$

see slide 10.13



$$B = \frac{\mu}{\mu_0} B_0$$

A-cross section of the core

$$L_1 L_2 = \frac{\mu N_1{}^2 A}{l} \frac{\mu N_2{}^2 A}{l} = M^2$$

$$M = \sqrt{L_1 L_2}$$

This is valid for two coils of identical length

Energy of a Magnetic Field in an Inductor

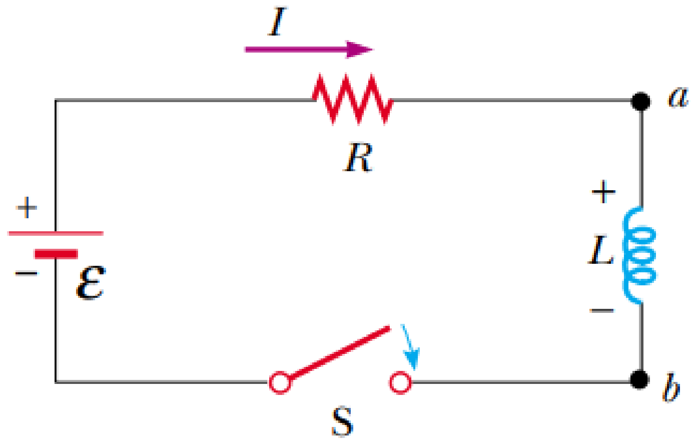
Because the *emf* induced in an inductor prevents a battery from establishing an instantaneous current, the battery must provide more energy than in a circuit without the inductor.

Part of the energy supplied by the battery appears as internal energy in the resistor. The remaining energy is stored in the magnetic field of the inductor.

Looking at this energy (in terms of rate) $$\mathrm{I}\,\varepsilon = \mathrm{I}^2 R + L\,\mathrm{I}\frac{d\mathrm{I}}{dt}$$

× I

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$



- $\mathrm{I}\,\varepsilon$ is the rate at which energy is being supplied by the battery.
- $\mathrm{I}^2 R$ is the rate at which the energy is being delivered to the resistor.
- Therefore, $LI\,(dI/dt)$ must be **the rate at which the energy is being "stored" in the magnetic field in the inductor**.

Let U_L denote the energy stored in the inductor at any time.

The rate at which the energy is stored is $$\frac{dU_L}{dt} = L\,\mathrm{I}\frac{d\mathrm{I}}{dt}$$

To find the total energy, integrate and $$U_L = L\int_0^{\mathrm{I}} \mathrm{I}\;\, d\mathrm{I} = \frac{1}{2}LI^2$$

Energy Stored in an Inductance

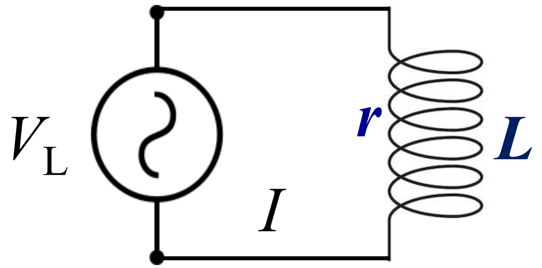
The induced current works against =>
The external source have to supply energy

$$\mathcal{E} = -L\frac{dI}{dt}$$

Where does the energy go? How much?

$$\mathcal{E} + r \cdot I = V_L$$ ***Kirkoff***

$$P = V_L\, I = -L\frac{dI}{dt}\, I + rI^2$$



r *is the resistance of the Inductor* ***L***

If r = 0

$$U_L = -\int P_L\, dt = L\int_0^I \frac{dI}{dt}\, I\, dt = \frac{1}{2}\, L\, I^2$$

- **This energy will be returned back when *I* → 0**

Energy is stored in the *B* field

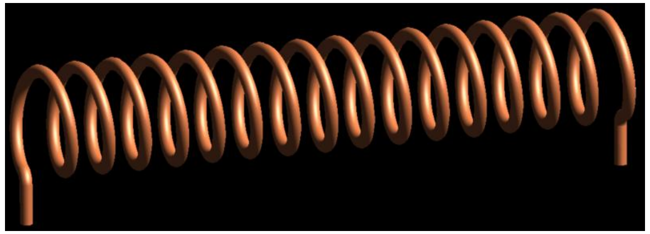
Similarity: compare with the electric field case confined in a capacitor C

$$U_C = \frac{1}{2}\, C\, V^2$$

Energy Storage in a Solenoid

For a solenoid: $B = \mu_0 n I \;\; (n = N/l)$.

$$E_L = \frac{1}{2} L I^2$$



$$\Longrightarrow \; I = B/\mu_0 n$$

The inductance

$$L = \frac{\mu_0 N^2 A}{l}$$

- N turns ;
- Length ℓ ;
- $N/\ell = n$;
- A – area ;

Therefore, the solenoid stores

Volume inside the solenoide

$$U_L = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{l} \left(\frac{Bl}{\mu_0 N} \right)^2 = \frac{1}{2} A l B^2/\mu_0$$

Energy density $u_L = U_L/V = \frac{1}{2} B^2/\mu_0$

stored in the volume of the solenoid

IT IS VALID IN GENERAL

This applies to any region in which a magnetic field exists (not just the solenoid).

Energy is Stored in Fields!

A resistor, inductor and capacitor all store energy through different mechanisms.

- Charged capacitor
  - Stores energy as electric potential energy
- Inductor
  - When it carries a current, stores energy as magnetic potential energy
- Resistor
  - Energy delivered is transformed into internal energy

- When a capacitor is charged, an electric field is created.
- The capacitor's energy is **stored in *E* field** with energy density

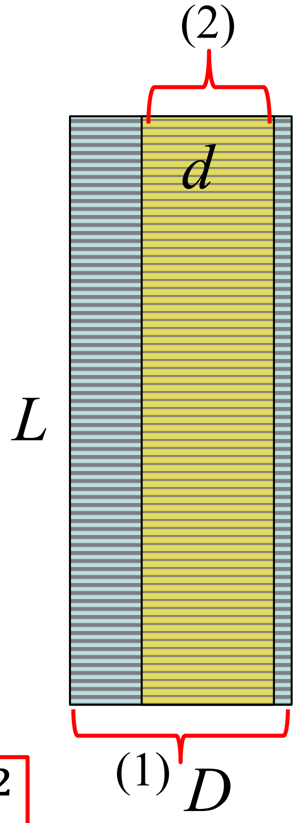
$$u_E = ½\, \varepsilon_0\, E^2$$

- When a current flows through an inductor, a magnetic field is created.
- The inductor's energy is **stored in *B* field** with energy density

$$u_B = ½\, B^2/\mu_0$$

Solving problem

A solenoid (2) of the diameter ***d*** is inserted into another solenoid (1) of the same length ***L***, but a larger diameter ***D***. The number of turns of wire, *N*, is the same in both solenoids.



What is M_{21}? Does $M_{21} = M_{12}$?

Solution:

$$M_{21} = -\frac{\varepsilon_2}{\partial I_1/\partial t} \quad B_1 = \mu_0 \cdot (N/L) \cdot I_1, \quad A_2 = 1/4 \cdot \pi d^2$$

Flux of the MF from the 1st solenoid through the 2nd:

$$\Phi_2 = A_2 N \cdot B_1 = A_2 N \cdot \mu_0 \cdot (N/L) \cdot I_1 \Rightarrow$$

$$\varepsilon_2 = -d\Phi_2/dt = -A_2 \cdot N \cdot \mu_0 \cdot (N/L) \cdot (dI_1/dt) \Rightarrow \quad \boxed{M_{21} = \mu_0 \frac{A_2 N^2}{L}}$$

In the same way one can show: $\Phi_1 = A_2 N \cdot B_2 = A_2 N \cdot \mu_0 \cdot (N/L) \cdot I_2 \Rightarrow$

$$\varepsilon_1 = -d\Phi_1/dt = -A_2 \cdot N \cdot \mu_0 \cdot (N/L) \cdot (dI_2/dt) \Rightarrow \quad M_{12} = M_{21}$$

Summary of last two Lectures

- Changing magnetic field produces an electric field (General form of Faraday's law):

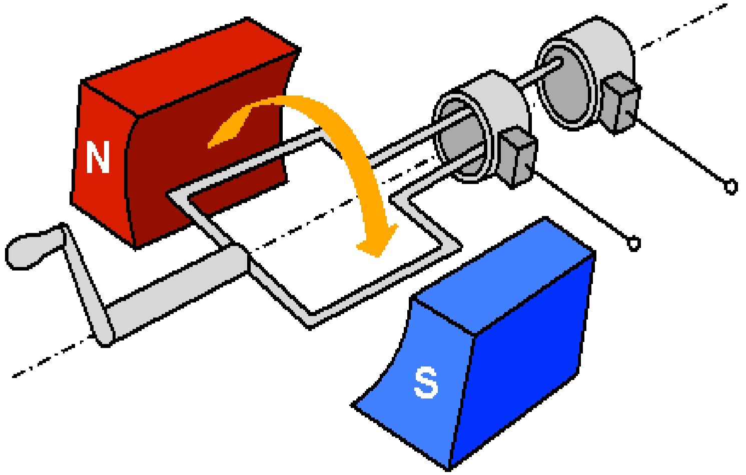
$$\oint \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

- Motor: electrical energy → mechanical
- Generator: mechanical → electrical

AC motor:

$$\varepsilon = \varepsilon_0 \sin(\omega t)$$

$$\varepsilon_0 = NBS\omega$$



Summary of last two Lectures

- **Self-inductance:**

$$\mathcal{E}_2 = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

$$L = \frac{Nd\Phi_B}{dI}$$

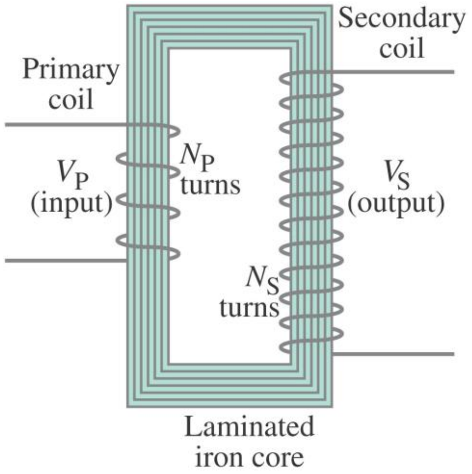
- **Mutual inductance:**

$$\mathcal{E}_2 = -M\frac{dI_1}{dt}$$

$$M_{21} = \frac{N_2 d\Phi_{21}}{dI_1}$$

Transformer:

$$\frac{V_p}{V_s} = \frac{N_p}{N_s}$$



RL Circuit, Introduction

A circuit element that has a large self-inductance is called an **inductor.**

The circuit symbol is 

We assume the self-inductance of the rest of the circuit is negligible compared to the inductor.

- However, even without a coil, a circuit will have some self-inductance.

RL Circuit

Effect of an Inductor in a Circuit

The inductance results in a back emf.

Therefore, the inductor in a circuit opposes changes in current in that circuit.

- The inductor attempts to keep the current the same way it was before the change occurred.
- The inductor can cause the circuit to be “sluggish” as it reacts to changes in the voltage.

RL Circuit

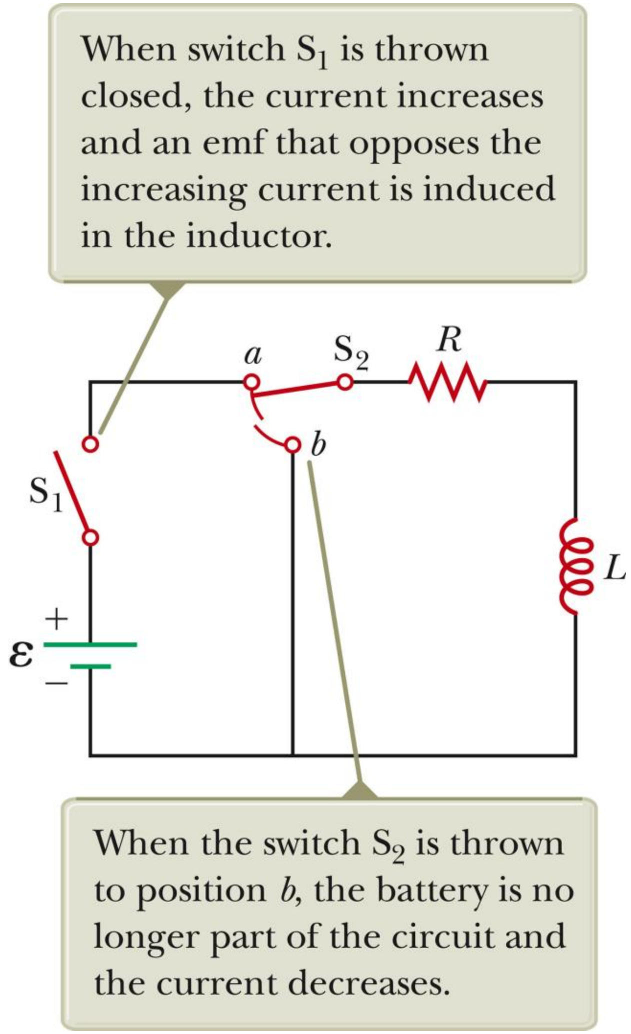
RL Circuit, Analysis

An *RL* circuit contains an inductor and a resistor.

Assume S_2 is connected to a

When switch S_1 is closed (at time t = 0), the current begins to increase.

At the same time, a back emf is induced in the inductor that opposes the original increasing current.



RL Circuit

RL Circuit, Analysis, cont.

Applying Kirchhoff's loop rule to the previous circuit in the clockwise direction gives

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

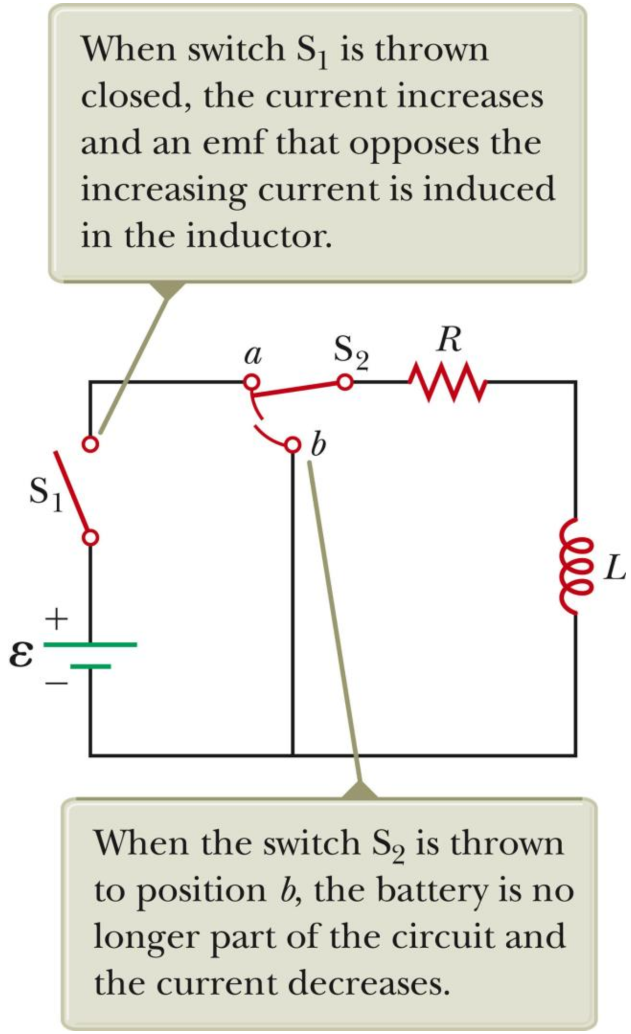
Looking at the current, we find

$$I = \frac{\varepsilon}{R}\left(1 - e^{-Rt/L}\right)$$

which is a first-order differential equation for I(t)

Notice its similarity to the equation for a capacitor and resistor in series (see RC Circuits).

Similarly, the solution can be found by making substitutions in the equations relating the capacitor to the inductor.



RL Circuit

RL Circuit, Analysis, Final

The inductor affects the current exponentially.

The current does not instantly increase to its final equilibrium value.

If there is no inductor, the exponential term goes to zero and the current would instantaneously reach its maximum value as expected.

- An inductor can act as a “surge protector” for sensitive electronic equipment that can be damaged by high currents.
- If equipment is plugged into a standard wall plug, a sudden “surge,” or increase, in voltage will normally cause a corresponding large change in current and damage the electronics.
- However, if there is an inductor in series with the voltage to the device, the sudden change in current produces an opposing emf preventing the current from reaching dangerous levels.

RL Circuit

RL Circuit, Current-Time Graph, Charging

The equilibrium value of the current is ε/R and is reached as *t* approaches infinity.

The current initially increases very rapidly.

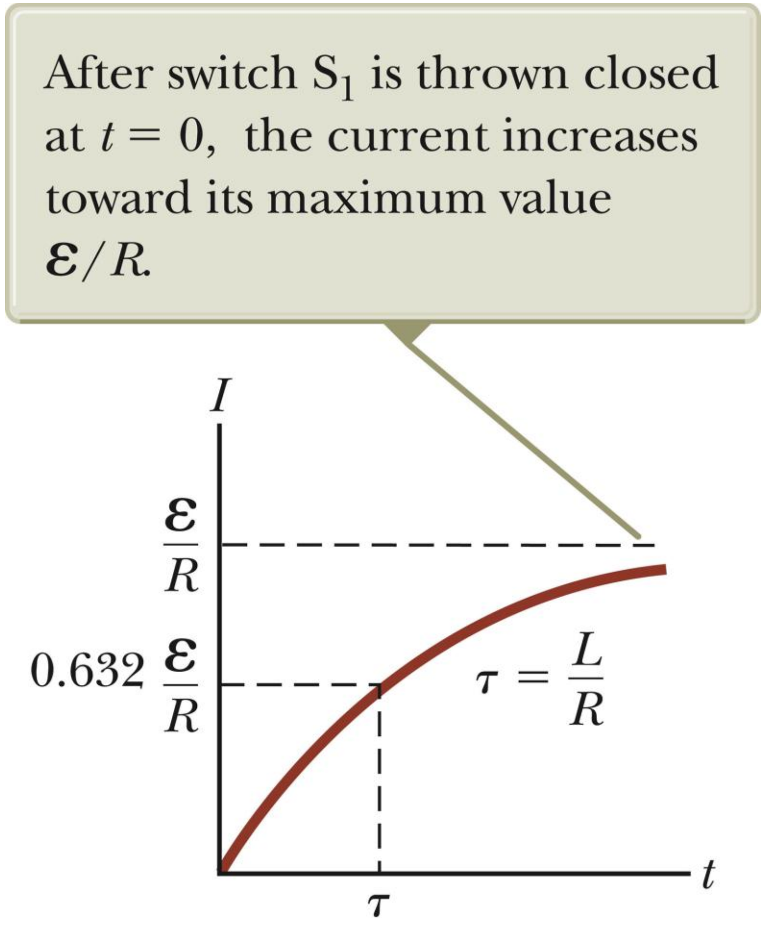
The current then gradually approaches the equilibrium value.

The expression for the current can also be expressed in terms of the time constant, τ, of the circuit.

$$I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

- where $\tau = L / R$

Physically, τ is the time required for the current to reach 63.2% of its maximum value.



RL Circuit

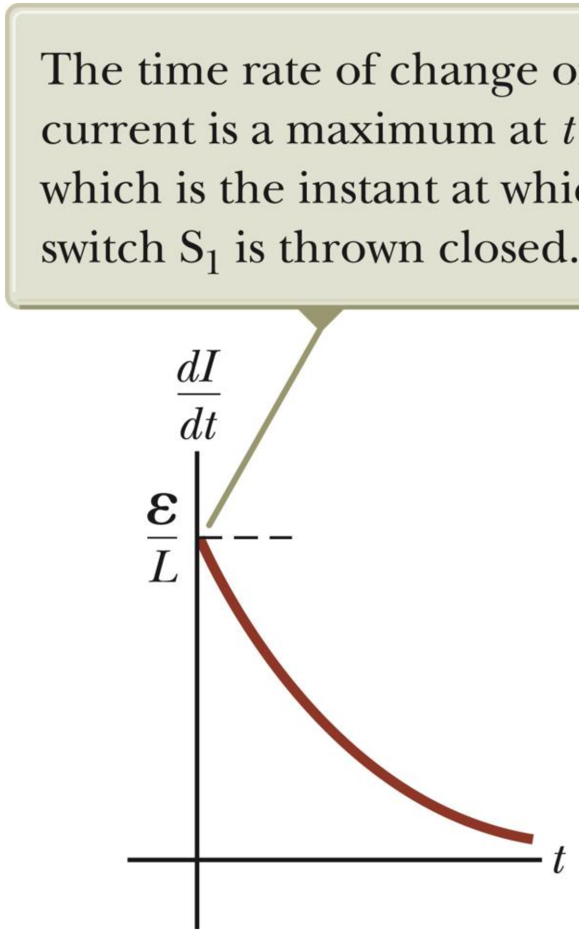
RL Circuit, Current-Time Graph, Discharging

The time rate of change of the current is a maximum at $t = 0$.

It falls off exponentially as t approaches infinity.

In general,

$$\frac{d\mathrm{I}}{dt} = \frac{\varepsilon}{L} e^{-\frac{t}{\tau}}$$



RL Circuit

RL Circuit Without A Battery

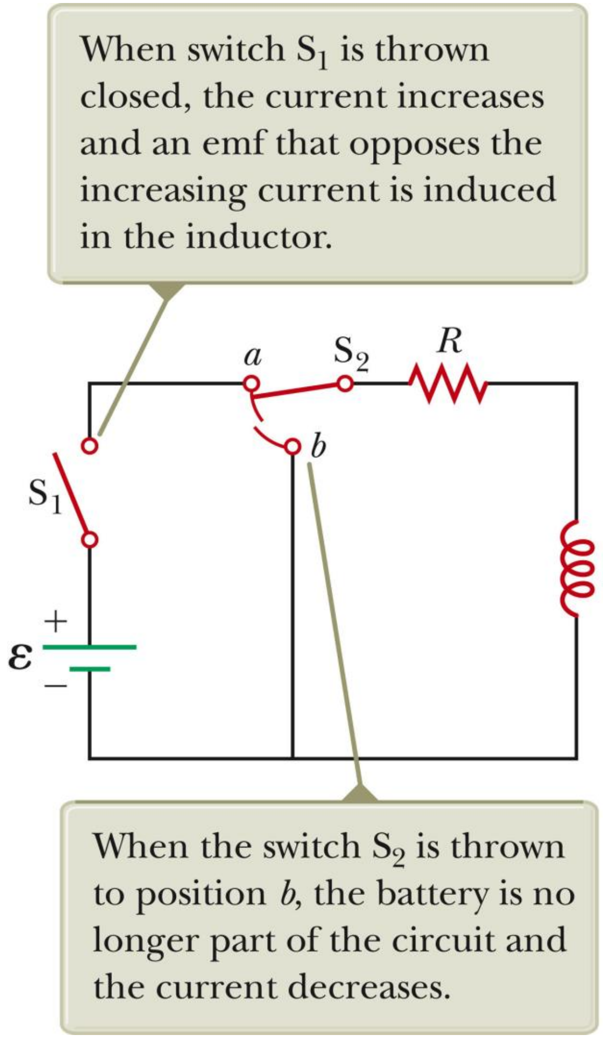
Now set S_2 to position b

The circuit now contains just the right hand loop .

The battery has been eliminated.

The expression for the current becomes

$$I = \frac{\varepsilon}{R} e^{-\frac{t}{\tau}} = I_i e^{-\frac{t}{\tau}}$$



Example: The Coaxial Cable

Coaxial cables are often used to connect electrical devices, such as your stereo system, and in receiving signals in TV cable systems.
Model a long coaxial cable as consisting of two thin concentric cylindrical conducting shells of radii *a* and *b* and length ℓ. The conducting shells carry the same current *I* in opposite directions.
Imagine that the inner conductor carries current to a device and that the outer one acts as a return path carrying the current back to the source.

The magnetic flux through a rectangle of width dr and length ℓ (along the cable), a distance r from the center, is

$$d\Phi_B = B(\ell\, dr) = \frac{\mu_o \mathrm{I}}{2\pi r} dr$$

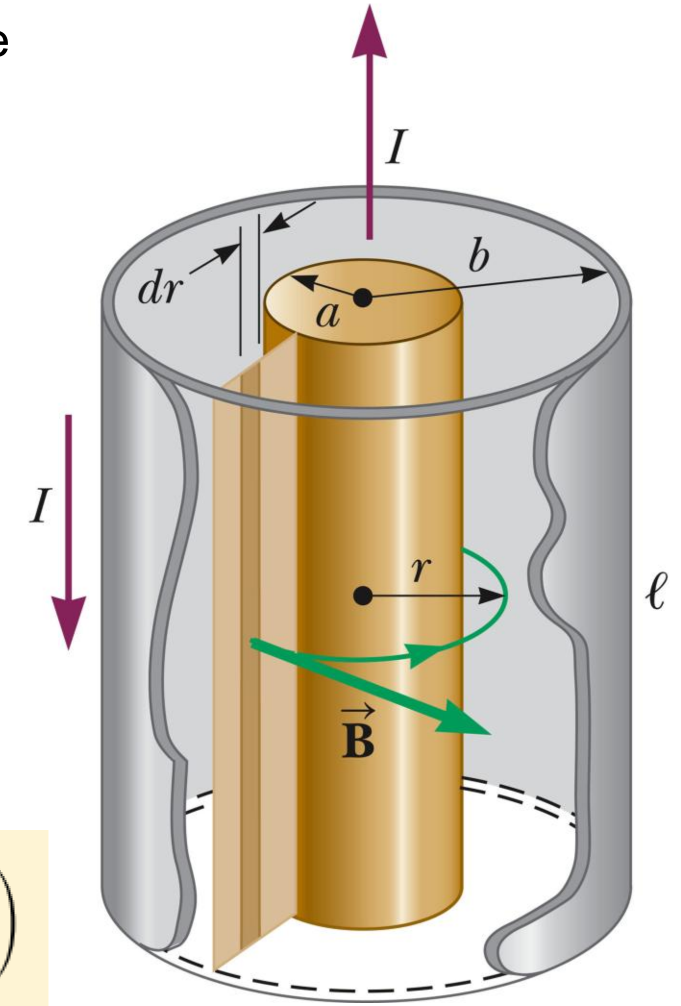
Calculate *L* of a length ℓ for the cable

The total flux is

$$\Phi_B = \int B\, dA = \int_a^b \frac{\mu_o \mathrm{I}\ell}{2\pi r} dr = \frac{\mu_o\, \mathrm{I}\, \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$

Therefore, *L* is

$$L = \frac{\Phi_B}{I} = \frac{\mu_o \ell}{2\pi} \ln\left(\frac{b}{a}\right)$$



To finalize the problem, note that the inductance increases if ℓ increases, if *b* increases, or if *a* decreases. This is consistent with our conceptualization —any of these changes increases the size of the loop represented by our radial slice and through which the magnetic field passes; this increases the inductance.

$$U = \frac{1}{2} L I^2 = \frac{\mu_0 \ell I^2}{4\pi} \ln\left(\frac{b}{a}\right)$$